Mathematical Analysis

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Chapter 3: Differential Calculus & Series Expansion

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Class Note Synopsis (Part 2)
B.Sc Semester 3
Subtopic: Function of a Single Variable

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Inverse function (Introduction), H.W, Limit f a Function (introduction) Appendix:

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♦ *This is Draft overview version of classnote* ♦ *November 17, 2021*

Instruction/ Suggestion:

This is Draft overview version of classnote. Such draft overview version of classnotes will be given time-to-time as a draft synopsis of the class discussions. Dear students, you should make your own "handwritten" classnote for your own future references & you are advised to write down in details in your own notebook and complete all home works (H.W) that will be given time-to-time (refer to class discussions for solutions and hints).

1 H.W. from previous class

Home work 1. 1. *Show that $f : \Re \to \Re$ defined by $f(x) = x^3$ is bijective function.*

**T/F *Check whether $f : \Re^+ \to \Re^+$ defined by $f(x) = x^2$ is bijective function.*

**T/F *Check whether $f(x) = [x]$ is bijective function.*

2. *Let $f : \Re \to \Re$ defined by $f(x) = x(x-1)(x-2)$. Then:*
   (a) *f is one-one but not onto.*
   (b) *f is not one-one but is onto.*
   (c) *f is neither one-one nor onto.*
   (d) *f is one-one and onto.*

3. *Check whether $f : \Re \to \Re$ defined by $f(x) = t(x)[t(x) + t(-x)]$ is ven function / odd function if*
   (a) *if f even function*
   (b) *if f odd function*

2 Inverse of Real valued function

an inverse function (or anti-function) is a function that "reverses" another function: if the function f applied to an input x gives an output y, then applying its inverse function g (denoted as f^{-1} to y gives the result x.

Definition 2.1 (Inverse of a function:). *Let f be a function whose domain is the set x, and whose codomain is the set Y. Then f is invertible if there exists a function g with domain Y and codomain X, with the property:*

$$f(x) = y \iff g(y) = x$$

Theorem 1 (Uniqueness of Inverse). *If f is invertible, then the function g is unique.*
which means that there is exactly one function g satisfying this property. Moreover, it also follows that the ranges of g and f equal their respective codomains (both bijection).
The function g is called the inverse of f, and is usually denoted as f^{-1}.

Example 2.1. *(1) To convert Fahrenheit to Celsius: $f(F) = (F - 32) \times 5/9$. The Inverse Function (Celsius back to Fahrenheit): $f^{-1}(C) = (C \times 9/5) + 32$*
*(2) $f(x) = x^2$ does not have an inverse but $\{x^2 | x \geq 0\}$ does have an inverse. (** Refer to class discussion)*

Home work 2.

1. *Show that, $(f^{-1})^{-1} = f$*
2. *If f^{-1} exists then show that f is bijective function.*
3. *The graph of $f(x)$ and $f^{-1}(x)$ are symmetric across the line $y = x$.*
4. *Find f^{-1} if $f(x) = \log\left(\frac{1+x}{1-x}\right)$, where $|x| < 1$*

3 Limit of a function

3.1 Recall !

In calculus, you have already learnt limit. Intuitively,

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number.

If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L.
(As x gets closer to a, $f(x)$ **gets closer and stays close** to L.) Symbolically, we express this idea as $\lim_{x \to a} f(x) = L$.

4 Analytical viewpoint of "Limit" of function

4.1 Previously (in 1 st Chapter) studied definition

Definition 4.1. *A function $f(x)$ is said to tend to a limit L as x tends to a (we write $\lim_{x \to a} f(x) = L$.) if :*

$$\forall \epsilon > 0, \ \exists \delta \ \ni |f(x) - L| < \epsilon \ \forall \ |x - a| < \delta$$

H.W.
Again Study & Write in notebook the followings: from Textbook (e.g., Apostol)

- $\epsilon - \delta$ Definition of limit of function when limit $L = +\infty$
- $\epsilon - \delta$ Definition of limit of function when limit $L = +\infty$
- $\epsilon - \delta$ Definition of limit of function for Right hand limit
- $\epsilon - \delta$ Definition of limit of function for Left hand limit
- Properties of limit (already studied in 11-12 syllabus)

Home work 3. *Using definition of limit, show that* $\lim_{x\to 0} x \sin \frac{1}{x} = 0$

H.W. hint: See any textbook for similar problem and its solution (e.g., Apostol). □

4.2 Sequential Definition of Limit of a function

We can combine some of the concepts that we have introduced before:

1. functions,
2. sequences,
3. topology

. If we have some function $f(x)$ and a given sequence a_n, then we can apply the function to each element of the sequence, resulting in a new sequence.
What we need is that if the original sequence converges to some number L, then the new sequence $f(a_n)$ should converge to $f(L)$, and if the original sequence diverges, the new one should diverge also.

Definition 4.2 (Limit of a Function (sequences version))**.** *A function f with domain $D \subset \Re$ converges to a limit L as x approaches a number c if $D - \{c\}$ is not empty and for* **any sequence** *$\{x_n\} \in D - \{c\}$ that converges to c, the sequence $\{f(x_n)\}$ converges to L. We write $\lim_{x\to c} f(x) = L$*

Usefulness of Sequential Definition of limit

Show that, $\lim_{x\to 0} \frac{|x|}{x}$ does not exists.
Proof. Let us consider two sequences $\{x_n\}, \{y_n\}$ where $x_n = 1/n \to 0$ and $y_n = -1/n \to 0, n \in \mathbb{N}$.
Then, $f(x_n) = 1 \to 1$ and $f(y_n) = -1 \to -1$ (trivially).
Therefore, $\lim_{x\to 0} \frac{|x|}{x}$ does not exists, as it is violating the sequential definition of limit. □

**** Refer to class discussion.

Home work 4. *Determine whether the following limits exists or not:*

1. $\lim_{x\to 0} \cos(1/x)$
2. $\lim_{x\to 0} [x]$

3. $\lim_{x \to 0} sgn(x)$

4. $\lim_{x \to 0} c1/x$

5. $\lim_{x \to 0} 1/x \cdot \sin(1/x)$

Proof. Hint for H.W Take suitably (** class discussion)
$x_n = \dfrac{1}{n+1}$, $y_n = \dfrac{-1}{n+1}$,
$x_n = \dfrac{2}{\pi(4n+1)}$, $y_n = \dfrac{1}{n\pi}$ □

Home work 5. *Show that,* $\lim_{x \to \infty} x \sin(x)$ *does not exists in* $\Re \cup \{+\infty\}$.

Home work 6 (Do this Experiment :)**.** *Consider the function f, where* $f(x) = 1$ *if* $x \leq 0$ *and* $f(x) = 2$ *if* $x > 0$. *The sequence* $\{1/n\}$ *converges to 0.*

1. *What happens to the sequence* $\{f(1/n)\}$ *?*

2. *The sequence* $\{3 + (-1)^n\}$ *is divergent. What happens to the sequence* $\{f(3 + (-1)^n)\}$ *?*

3. *The sequence* $\{(-1)^n/n\}$ *converges to* 0. *What happens to the sequence* $\{f((-1)^n/n)\}$ *?*

5 References

1. Apostol: Mathematical Analysis

2. Wekepedia references